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EFFECTS OF THE SOLAR AND LUNAR TIDES ON THE MOTION OF AN ARTIFICIAL EARTH SATELLITE

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SUMMARY

The series describing the perturbations in the orbital elements of an artificial satellite due to solar and lunar tides are presented in a form convenient for computation. In addition, the potential function and its gradient are given in terms of rectangular coordinates.

CONTENTS

LIST OF SYMBOLS

A - coefficient of trigonometric term in the determining function

A_p - coefficient of trigonometric term in the disturbing function

$A_{\nu 0}$ - factor in the coefficient of the trigonometric term of the determining function depending only on the inclination of the satellite orbit plane to the equator

a - semi-major axis of the satellite

$C_{q' i' \nu}$ - coefficient of trigonometric term depending on the semi-major axis, the eccentricity of the satellite, and the mean motion of the argument of the trigonometric term

c - cosine of half the angle of inclination of satellite orbit plane to the equator

e - eccentricity of the satellite

G - Delaunay variable conjugate to argument of perigee of satellite

g - argument of perigee of satellite

H - Delaunay variable conjugate to right ascension of node of satellite

h - Delaunay symbol for right ascension of the node - same as Ω

I - inclination of satellite orbit plane to the equator

I' - inclination of orbit of disturbing body to the equator

i' - index associated with mean anomaly of disturbing body

J_2 - second zonal harmonic of earth

k_2 - Love number

L - Delaunay variable conjugate to mean anomaly of satellite

ℓ - mean anomaly of satellite

ℓ'^* - mean anomaly of disturbing body, including phase angle due to tidal friction

$\ell_{☾}$ - mean anomaly of the moon

$\ell_{\odot}$ - mean anomaly of the sun

m' - ratio of mass of disturbing body to the sum of the mass of the disturbing body and the mass of the earth

n - mean motion of the satellite

n' - mean motion of the disturbing body

q' - index associated with argument of perigee of the disturbing body

R - mean radius of earth

R_p - periodic part of the disturbing function

r - geocentric distance of the satellite

$r_{☾}^{*}$, $r_{\odot}^{*}$ - geocentric distance of fictitious moon (sun)

S - determining function for solar and lunar tides

$T_{q', i', \nu}$ - coefficient of trigonometric term in the determining function depending on elements of disturbing body only

α - argument in the disturbing function

$\dot{\alpha}$ - mean motion of α

$\beta_{☾}^{*}$, $\beta_{\odot}^{*}$ - geocentric angles between r and $r_{☾}^{*}$, $r_{\odot}^{*}$

γ - sine of one half the angle of inclination of satellite orbit to the equator

ϵ - phase lag due to tides associated with the right ascension

κ_1 - phase angle associated with mean anomaly of the moon due to tidal friction

κ_2 - phase angle associated with mean anomaly of the sun due to tidal friction

ν - index associated with right ascension of the node

Ω - right ascension of the satellite

$\Omega_{☾}$ - right ascension of the moon

$\dot{\Omega}$ - mean motion of Ω

$\dot{\Omega}_{☾}$ - mean motion of $\Omega_{☾}$

ω' - argument of perigee of disturbing body

θ - difference of right ascension of the satellite and the disturbing body, including the phase lag ϵ

$\dot{\theta}$ - mean motion of θ

μ - product of gravitational constant and mass of the earth

k^2 - gravitational constant

EFFECTS OF THE SOLAR AND LUNAR TIDES ON THE MOTION OF AN ARTIFICIAL EARTH SATELLITE

INTRODUCTION

In References 1 and 2, it has been shown that solar and lunar tides may cause perturbations in the trajectories of satellites; also, the disturbing potential which describes the variation of the earth's geoid with time has been discussed.

In this report, a determining function corresponding to the disturbing function for the tides is introduced, from which perturbations in the orbital elements are derived. The perturbations are given in a form suitable for programming.

It is also useful to have available the formulation of the potential function and its gradient in terms of rectangular coordinates. These are presented in the last section.

THE DETERMINING FUNCTION

The periodic part of the disturbing function is given in Reference 1 in the form

$$R_p = \sum A_p \cos \alpha \ , \tag{1}$$

where A_p is a function of the coordinate elements of the satellite and the disturbing body, and α is defined by

$$\alpha = i' \ell'^{*} + q' \omega' + \nu\theta \ . \tag{2}$$

Here, q', i', and ν are integers defining the indices of the summation (1). The index q' takes on only two values, zero and two, and thus generates Tables 1 and 2 at the end of the report. When q' equals zero, i' takes the values -1, 0, and 1, while ν takes on the values 0, 1, and 2. When q' equals two, i' assumes the values -1, -2, -3, and -4, while ν varies from -2 to +2. The case when all three integers are zero simultaneously leads to secular terms and is discussed in Reference 1.

The angular variables in Equation (2) are defined by

$$\begin{aligned} \ell'^{*} &= \ell_{\text{☾}} - \kappa_1 \text{ for the moon} \qquad & \theta &= \Omega - \Omega_{\text{☾}} - \epsilon \text{ for the moon} \\ \ell'^{*} &= \ell_{\odot} - \kappa_2 \text{ for the sun} \qquad & \theta &= \Omega - \epsilon \text{ for the sun} \ , \end{aligned} \tag{3}$$

where

$\ell_{☾}$ = mean anomaly of the moon

$\ell_{\odot}$ = mean anomaly of the sun

$\Omega_{☾}$ = right ascension of the node of the moon

Ω = right ascension of the node of the satellite

ω' = argument of perigee of the disturbing body

κ_1, κ_2, ϵ are phase angles due to tidal friction.

From Equation (1), we find by the methods of Reference 3, for example, that the first order determining function S is given by

$$S = \sum A \sin \alpha \ , \tag{4}$$

where

$A = A_p/\dot{\alpha}$

$\dot{\alpha}$ = mean motion of α.

The quantity A in Equation (4) may be decomposed into the factors

$$A = k_2 R^5 n'^2 m' T_{q'i'\nu} A_{\nu 0} C_{q'i'\nu} \ , \tag{5}$$

where

k_2 = Love number (approximately .3)

R = mean radius of the earth

n' = mean motion of the disturbing body

m' = ratio of the mass of the disturbing body to the sum of the mass of the disturbing body and the mass of the earth.

The function $T_{q'i'\nu}$ is independent of the elements of the satellite and is given by the formula

$$T_{q'i'\nu} = A'_{\nu q'} B'_{i'q'} \ . \tag{6}$$

$A'_{\nu q'}$ and $B'_{i'q'}$ are functions of the elements of the disturbing body only and are given in Table 1 for $q' = 0$ and Table 2 for $q' = 2$.

$A_{\nu 0}$ is a function of γ only, where

$$\gamma = \sin \frac{I}{2} \tag{7}$$

and I is the inclination of the orbit plane to the earth's equator. $A_{\nu 0}$ is given in Tables 1 and 2 for the appropriate ranges of the index ν. $C_{q'i'\nu}$ can be written explicitly as

$$C_{q'i'\nu} = \frac{1}{a^3 \left(1 - e^2\right)^{3/2} \dot{\alpha}} , \tag{8}$$

where a and e are the semi-major axis and eccentricity of the satellite. The motion $\dot{\alpha}$ is given by

$$\dot{\alpha} = i' n' + q' \dot{\omega}' + \nu \dot{\theta} , \tag{9}$$

where n', $\dot{\omega}'$, and $\dot{\theta}$ are the mean motions of ℓ'^*, ω', and θ defined previously. Also,

$$\begin{aligned} \dot{\theta} &= \dot{\Omega} - \dot{\Omega}_{☾} \text{ for the moon} \\ \dot{\theta} &= \dot{\Omega} \text{ for the sun} \end{aligned} \tag{10}$$

$$\dot{\Omega} = -\frac{3}{2} \frac{n J_2 \cos I}{a^2 \left(1 - e^2\right)^2} , \tag{11}$$

where

J_2 = second zonal harmonic of earth's potential

n = mean motion of the satellite.

THE PERTURBATIONS IN THE ELEMENTS

If we introduce the Delaunay variables:

$L = (\mu a)^{1/2}$

$G = L\left(1 - e^2\right)^{1/2}$

$H = G \cos I$

ℓ = mean anomaly of the satellite

g = argument of perigee of the satellite

$h = \Omega$ = right ascension of the node of the satellite,

where μ = product of the gravitational constant and the mass of the satellite, we can derive the perturbations in the elements from the determining function defined by Equation 4.

From the theory of canonical variables, and since the determining function S is independent of ℓ and g, the Delaunay variables L and G (and consequently a and e) are not affected by the solar or lunar tides. The perturbations in the remaining elements are given by

$$\delta H = -n a^2 \left(1-e^2\right)^{1/2} \sin I\, \delta I = +k_2 R^5 n'^2 m' \sum \nu T_{q'i'\nu} A_{\nu 0} C_{q'i'\nu} \cos \alpha$$

$$\begin{aligned}
\delta \ell &= -\frac{\partial S}{\partial L} = -k_2 R^5 n'^2 m' \sum T_{q'i'\nu} \left[A_{\nu 0} \frac{\partial C_{q'i'\nu}}{\partial L} + \frac{\partial A_{\nu 0}}{\partial L} C_{q'i'\nu} \right] \sin \alpha \\
\delta g &= -\frac{\partial S}{\partial G} = -k_2 R^5 n'^2 m' \sum T_{q'i'\nu} \left[A_{\nu 0} \frac{\partial C_{q'i'\nu}}{\partial G} + \frac{\partial A_{\nu 0}}{\partial G} C_{q'i'\nu} \right] \sin \alpha \\
\delta h &= -\frac{\partial S}{\partial H} = -k_2 R^5 n'^2 m' \sum T_{q'i'\nu} \left[A_{\nu 0} \frac{\partial C_{q'i'\nu}}{\partial H} + \frac{\partial A_{\nu 0}}{\partial H} C_{q'i'\nu} \right] \sin \alpha \quad .
\end{aligned} \tag{12}$$

In order to facilitate deriving differential coefficients, it is useful to note that Equation (11) in terms of Delaunay variables becomes

$$\dot{\Omega} = -\frac{3}{2} \frac{\mu^4 J_2 H}{L^3 G^5} \quad . \tag{13}$$

From Equations (2), (5), and (13) we find

$$\begin{aligned}
\frac{\partial \dot{\alpha}}{\partial L} &= \nu \frac{\partial \dot{\Omega}}{\partial L} = -\frac{3\nu}{n a^2} \dot{\Omega} \\
\frac{\partial \dot{\alpha}}{\partial G} &= \nu \frac{\partial \dot{\Omega}}{\partial G} = -\frac{5\nu}{n a^2 \left(1-e^2\right)^{1/2}} \dot{\Omega} \\
\frac{\partial \dot{\alpha}}{\partial H} &= \nu \frac{\partial \dot{\Omega}}{\partial H} = \frac{\nu}{n a^2 \left(1-e^2\right)^{1/2} \cos I} \dot{\Omega} \quad .
\end{aligned} \tag{14}$$

From the definition of $C_{q'i'\nu}$ given in (8) and the Equations (14), we find

$$\begin{aligned}
\frac{\partial C_{q'i'\nu}}{\partial L} &= -\frac{3 C_{q'i'\nu}}{n a^2} \left(1 - \nu \frac{\dot{\Omega}}{\dot{\alpha}}\right) \\
\frac{\partial C_{q'i'\nu}}{\partial G} &= -\frac{C_{q'i'\nu}}{n a^2 \left(1-e^2\right)^{1/2}} \left(3 - 5\nu \frac{\dot{\Omega}}{\dot{\alpha}}\right) \\
\frac{\partial C_{q'i'\nu}}{\partial H} &= -\frac{C_{q'i'\nu}}{n a^2 \left(1-e^2\right)^{1/2} \cos I} \nu \frac{\dot{\Omega}}{\dot{\alpha}} \quad .
\end{aligned} \tag{15}$$

Furthermore, since $A_{\nu 0}$ is a function of γ only, we have

$$
\begin{aligned}
\frac{\partial A_{\nu 0}}{\partial L} &= 0 \\
\frac{\partial A_{\nu 0}}{\partial G} &= \frac{\left(c^2 - \gamma^2\right)}{4n\, a^2 \left(1 - e^2\right)^{1/2} \gamma} \frac{d A_{\nu 0}}{d\gamma} \\
\frac{\partial A_{\nu 0}}{\partial H} &= - \frac{1}{4n\, a^2 \left(1 - e^2\right)^{1/2} \gamma} \frac{d A_{\nu 0}}{d\gamma} ,
\end{aligned}
\tag{16}
$$

where $c = \cos I/2$.

Combining the above results, the perturbations in the elements can be written in the form

$$
\begin{aligned}
\delta I &= - \frac{F}{2c\gamma} \sum T_{q'i'\nu} A_{\nu 0} \frac{\nu}{\dot{\alpha}} \cos \alpha \\
\delta \ell &= F\left(1 - e^2\right)^{1/2} \sum \frac{3}{\dot{\alpha}} \left(1 - \nu \frac{\dot{\Omega}}{\dot{\alpha}}\right) T_{q'i'\nu} A_{\nu 0} \sin \alpha \\
\delta g &= F \sum \left[\left(3 - 5\nu \frac{\dot{\Omega}}{\dot{\alpha}}\right) T_{q'i'\nu} A_{\nu 0} - \frac{c^2 - \gamma^2}{4\gamma} T_{q'i'\nu} \frac{d A_{\nu 0}}{d\gamma}\right] \frac{\sin \alpha}{\dot{\alpha}} \\
\delta h &= F \sum \left[\frac{\nu}{c^2 - \gamma^2} \frac{\dot{\Omega}}{\dot{\alpha}} T_{q'i'\nu} A_{\nu 0} + \frac{1}{4\gamma} T_{q'i'\nu} \frac{d A_{\nu 0}}{d\gamma}\right] \frac{\sin \alpha}{\dot{\alpha}} ,
\end{aligned}
\tag{17}
$$

where F is the constant introduced in Reference 1 and is given by

$$
F = \frac{k_2 R^5 n'^2 m'}{n\, a^5 \left(1 - e^2\right)^2} . \tag{18}
$$

THE DISTURBING FUNCTION AND ITS GRADIENT IN RECTANGULAR COORDINATES

Neglecting the parallactic part, the disturbing function due to tidal effects is

$$
U^* = \frac{k^2 R^5 k_2}{2r^3} \left\{ \frac{m_{☾}}{r_{☾}^{*3}} \left(3 \cos^2 \beta_{☾}^* - 1\right) + \frac{m_{\odot}}{r_{\odot}^{*3}} \left(3 \cos^2 \beta_{\odot}^* - 1\right) \right\} .
$$

With respect to an orthogonal, earth-centered inertial coordinate system where the z-axis coincides with the earth's axis of rotation, let

$$\vec{r} = (x, y, z)$$

$$\vec{r}_{☾}^{\,*} = \left(x_{☾}^*, y_{☾}^*, z_{☾}^*\right)$$

$$\vec{r}_{\odot}^{\,*} = \left(x_{\odot}^*, y_{\odot}^*, z_{\odot}^*\right) \quad .$$

Then,

$$\cos \beta_{☾}^* = \frac{\vec{r} \cdot \vec{r}_{☾}^{\,*}}{r r_{☾}^*} = \frac{x x_{☾}^* + y y_{☾}^* + z z_{☾}^*}{\sqrt{x^2 + y^2 + z^2}\sqrt{x_{☾}^{*2} + y_{☾}^{*2} + z_{☾}^{*2}}}$$

$$\cos \beta_{\odot}^* = \frac{\vec{r} \cdot \vec{r}_{\odot}^{\,*}}{r r_{\odot}^*} = \frac{x x_{\odot}^* + y y_{\odot}^* + z z_{\odot}^*}{\sqrt{x^2 + y^2 + z^2}\sqrt{x_{\odot}^{*2} + y_{\odot}^{*2} + z_{\odot}^{*2}}} \quad .$$

Thus,

$$\frac{\partial U^*}{\partial x} = \frac{3k^2R^5 k_2}{2r^5}\left\{\frac{m_{☾}}{r_{☾}^{*3}}\left[\frac{\vec{r} \cdot \vec{r}_{☾}^{\,*}}{r^2 r_{☾}^{*2}}\left(2x_{☾}^* r^2 - 5x\left(\vec{r} \cdot \vec{r}_{☾}^{\,*}\right)\right) + x\right] + \frac{m_{\odot}}{r_{\odot}^{*3}}\left[\frac{\vec{r} \cdot \vec{r}_{\odot}^{\,*}}{r^2 r_{\odot}^{*2}}\left(2x_{\odot}^* r^2 - 5x\left(\vec{r} \cdot \vec{r}_{\odot}^{\,*}\right)\right) + x\right]\right\}$$

$$\frac{\partial U^*}{\partial y} = \frac{3k^2R^5 k_2}{2r^5}\left\{\frac{m_{☾}}{r_{☾}^{*3}}\left[\frac{\vec{r} \cdot \vec{r}_{☾}^{\,*}}{r^2 r_{☾}^{*2}}\left(2y_{☾}^* r^2 - 5y\left(\vec{r} \cdot \vec{r}_{☾}^{\,*}\right)\right) + y\right] + \frac{m_{\odot}}{r_{\odot}^{*3}}\left[\frac{\vec{r} \cdot \vec{r}_{\odot}^{\,*}}{r^2 r_{\odot}^{*2}}\left(2y_{\odot}^* r^2 - 5y\left(\vec{r} \cdot \vec{r}_{\odot}^{\,*}\right)\right) + y\right]\right\}$$

$$\frac{\partial U^*}{\partial z} = \frac{3k^2R^5 k_2}{2r^5}\left\{\frac{m_{☾}}{r_{☾}^{*3}}\left[\frac{\vec{r} \cdot \vec{r}_{☾}^{\,*}}{r^2 r_{☾}^{*2}}\left(2z_{☾}^* r^2 - 5z\left(\vec{r} \cdot \vec{r}_{☾}^{\,*}\right)\right) + z\right] + \frac{m_{\odot}}{r_{\odot}^{*3}}\left[\frac{\vec{r} \cdot \vec{r}_{\odot}^{\,*}}{r^2 r_{\odot}^{*2}}\left(2z_{\odot}^* r^2 - 5z\left(\vec{r} \cdot \vec{r}_{\odot}^{\,*}\right)\right) + z\right]\right\} .$$

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Table 1

($q' = 0$)

$A'_{\nu q'}$	$A_{\nu 0}$	$\dfrac{d A_{\nu 0}}{d\gamma}$	ν
$\frac{1}{4}\left(1 - 6\gamma'^2 + 6\gamma'^4\right)$	$1 - 6\gamma^2 + 6\gamma^4$	$-12\gamma\left(1 - 2\gamma^2\right)$	0
$3c'\gamma'\left(1 - 2\gamma'^2\right)$	$c\gamma\left(1 - 2\gamma^2\right)$	$c^{-1}\left(1 - 8\gamma^2 + 8\gamma^4\right)$	1
$3c'^2\gamma'^2$	$c^2\gamma^2$	$2\gamma\left(1 - 2\gamma^2\right)$	2
	$B'_{i'q'}$		i'
	$\frac{3}{2}e'$		-1
	1		0
	$\frac{3}{2}e'$		1

Table 2

($q' = 2$)

$A'_{\nu q'}$	$A_{\nu 0}$	$\dfrac{d A_{\nu 0}}{d\gamma}$	ν
$\frac{3}{2}c'^4$	$e^2\gamma^2$	$2\gamma\left(1 - 2\gamma^2\right)$	-2
$-3c'^3\gamma'$	$c\gamma\left(1 - 2\gamma^2\right)$	$c^{-1}\left(1 - 8\gamma^2 + 8\gamma^4\right)$	-1
$\frac{3}{2}c'^2\gamma'^2$	$1 - 6\gamma^2 + 6\gamma^4$	$-12\gamma\left(1 - 2\gamma^2\right)$	0
$3c'\gamma'^3$	$c\gamma\left(1 - 2\gamma^2\right)$	$c^{-1}\left(1 - 8\gamma^2 + 8\gamma^4\right)$	1
$\frac{3}{2}\gamma'^4$	$c^2\gamma^2$	$2\gamma\left(1 - 2\gamma^2\right)$	2
	$B'_{i'q'}$		i'
	$-\frac{e'}{2}$		1
	1		2
	$\frac{7}{2}e'$		3